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The run transform

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ABSTRACT

We consider the transform from sequences to triangular arrays defined in terms of generating functions by $f(x) \to \frac{1-x}{1-xy} f\left(\frac{x(1-x)}{1-xy}\right)$. We establish a criterion for the transform of a nonnegative sequence to be nonnegative, and we show the transform counts certain classes of lattice paths by number of the so-called pyramid ascents and certain classes of partitions into sets of lists (blocks) by number of blocks that consist of increasing consecutive integers.

1. Introduction

We investigate the transform Φ defined on formal power series $f(x)$ by

$$\Phi\big(f(x)\big) := \frac{1-x}{1-xy} f\left(\frac{x(1-x)}{1-xy}\right).$$

Following Herbert Wilf's dictum, "A generating function is a clothesline on which we hang up a sequence of numbers for display" [9, Chapter 1], we will use sequences/arrays and their generating functions interchangeably. Thus the transform Φ is also defined for sequences $(a_n)_{n\geq 0}$. It turns out that the transform is closely related to the Catalan numbers and there is a nice combinatorial interpretation for the transform of the size-counting sequence for various classes of partitions into sets of lists (blocks) and various classes of lattice paths of upsteps U, flatsteps F, and downsteps D. In the former case, the transform counts partitions by number of runs, where a *run*, also known as an *adjacent block* [1], is a block that consists of increasing consecutive integers. In the latter case it counts lattice paths by number of pyramid ascents, where an *ascent* is a maximal subpath of the form U^k, $k \geq 1$, a *pyramid* is a maximal subpath of the form U^kD^k, $k \geq 1$, and a *pyramid ascent* is an ascent that is the first half of a pyramid. For example, among the four ascents of $UUDUUDUUDDDUDD$, only the last two (UU and U) are pyramid ascents.

Because of the interpretation in terms of runs, and for brevity, we will call Φ the *run transform*.

In Section 2, we review the Catalan numbers and two of their interpretations, and in Section 3 we establish some basic properties of the run transform and give a criterion for the run transform of a nonnegative sequence to also be nonnegative. Section 4 gives interpretations of the run transform of the Catalan numbers in terms of both Dyck paths and noncrossing partitions, the basis for subsequent generalizations. Section 5 generalizes to paths of j-upsteps (j, j) and downsteps $(1, -1)$. Section 6 recalls the notion of a set-of-lists partition, s-partition for short, introduces the notion of a run-closed family of s-partitions and proves that if $f(x)$ is the generating function by size of a run-closed family $\mathcal{F}$ of s-partitions, then the run

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transform of $f(x)$ counts $\mathcal{F}$ by size and number of runs. Section 7 generalizes this result. Section 8 considers the transform for paths of upsteps, flatsteps, and downsteps and Section 9 presents a conjecture.

Sequences in The On-Line Encyclopedia of Integer Sequences (OEIS) [8], are referred to by their six-digit A-numbers.

2. Review of the Catalan numbers, Dyck paths, and noncrossing partitions

The Catalan numbers (sequence A000108 in OEIS) are intimately related to the run transform Φ; so let us recall some facts and fix some notation for them and for two of their interpretations. The generating function for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n-1}$ is $C(x) = (1-\sqrt{1-4x})/(2x)$. The Catalan convolution matrix is defined by $C = \left(\binom{2j-i}{j-i} - \binom{2j-i}{j-i-1}\right)_{i,j\geq 0}$ and its inverse is given by $C^{-1} = \left((-1)^{j-i}\binom{i+1}{j-i}\right)_{i,j\geq 0}$ [5, p. 137]. The first few rows and columns are shown.

$$C = \begin{pmatrix} 1 & 1 & 2 & 5 & 14 & 42 & 132 & \dots \\ & 1 & 2 & 5 & 14 & 42 & 132 & \dots \\ & & 1 & 3 & 9 & 28 & 90 & \dots \\ & & & 1 & 4 & 14 & 48 & \dots \\ & & & & 1 & 5 & 20 & \dots \\ & & & & & 1 & 6 & \dots \\ & & & & & & 1 & \dots \\ & & & & & & & \ddots \end{pmatrix}, \qquad C^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & \dots \\ & 1 & -2 & 1 & 0 & 0 & 0 & \dots \\ & & 1 & -3 & 3 & -1 & 0 & \dots \\ & & & 1 & -4 & 6 & -4 & \dots \\ & & & & 1 & -5 & 10 & \dots \\ & & & & & 1 & -6 & \dots \\ & & & & & & 1 & \dots \\ & & & & & & & \ddots \end{pmatrix}.$$

Thus the top row (row 0) of C, the Catalan numbers, has generating function $C(x)$. It is well known that the entries of C count nonnegative lattice paths. Specifically, define a U–D path to be a lattice path of *upsteps* $U = (1, 1)$ and *downsteps* $D = (1, -1)$ (not necessarily starting at the origin). Let $\mathcal{N}_{mn}$ $(=\mathcal{N}_{m,n})$ denote the set of U–D paths consisting of n upsteps and $m + n$ downsteps that never dip below ground level, the horizontal line through the terminal vertex. The size of a U–D path is its number of downsteps. Then $|\mathcal{N}_{mn}| = C_{mn}$. Set $\mathcal{N}_m := \bigcup_{n\geq 0} \mathcal{N}_{mn}$. A *Dyck* path is a member of $\mathcal{N}_0$. It is also well known (see, e.g., [10, Remark 5] or [7]) that row m of C, starting at the diagonal entry, is the $(m + 1)$-fold convolution of the top row. Hence, the generating function for $\mathcal{N}_m$ by size is $x^m C(x)^{m+1}$, and, with $\mathbf{x}$ defined to be the column vector $(1, x, x^2, \dots)^t$ where the superscript t denotes transpose, we have the matrix–vector product

$$C\,\mathbf{x} = (C(x), xC(x)^2, x^2C(x)^3, \dots)^t. \tag{1}$$

We define $C(x, y)$ to be $\Phi\big(C(x)\big)$. Thus

$$C(x, y) = \frac{1-\sqrt{1-4\,\frac{x(1-x)}{1-xy}}}{2x}. \tag{2}$$

The *matching step* of a given step in a Dyck path is the other end-step of the shortest Dyck subpath containing the given step as an end-step. Ascents (and pyramid ascents) were defined in the Introduction and, analogously, a *descent* is a maximal subpath of the form $D^k,\ k \geq 1$.

A nonempty Dyck path decomposes (at its returns to ground level) into *components*, each of which is a *primitive* Dyck path—a nonempty Dyck path whose only return to ground level is at the end.

There is a well known bijection from Dyck paths to noncrossing partitions, whose origin we have been unable to trace. Traverse the Dyck path from left to right and number the down steps from 1 to n. Give the same labels to the matching up steps. The numbers on the ascents form the blocks of the partition. We will call this the standard bijection. Under the standard bijection, pyramid ascents in the Dyck path become runs in the noncrossing partition.

3. Basic properties and a criterion for nonnegativity

We will use $F(x, y)$ for $\Phi\big(f(x)\big)$ to show the dependency on both variables. Clearly, $F(x, 1) = f(x)$ and so the row sums of the run transform give the original sequence. The run transform Φ is linear,

$$\Phi\big(\alpha f(x) + \beta g(x)\big) = \alpha\Phi\big(f(x)\big) + \beta\Phi\big(g(x)\big),$$

and has a multiplicativity property,

$$\Phi\big(xf(x)g(x)\big) = x\Phi\big(f(x)\big)\Phi\big(g(x)\big).$$

More generally,

$$\Phi\big(x^{i-1}f_1(x)f_2(x)\cdots f_i(x)\big) = x^{i-1}\Phi\big(f_1(x)\big)\Phi\big(f_2(x)\big)\cdots\Phi\big(f_i(x)\big).$$

In particular, for $i = k + 1$ and $f = f_1 = f_2 = \cdots$,

$$\Phi\big(x^k f(x)^{k+1}\big) = x^k\Phi\big(f(x)\big)^{k+1}.$$

From this fact, together with linearity, we obtain the following lemma.

Lemma 1. *For an arbitrary sequence $(a_k)_{k\geq 0}$ and formal power series $f(x)$,*

$$\Phi\left(\sum_{k\geq 0} a_k x^k f(x)^{k+1}\right) = \sum_{k\geq 0} a_k x^k \Phi\big(f(x)\big)^{k+1}.$$

Proposition 2. *Let $\mathbf{a} = (a_k)_{k\geq 0}$ be an arbitrary sequence. Then its run transform is*

$$\sum_{k\geq 0} b_k x^k C(x,y)^{k+1},$$

where $\mathbf{b} = (b_k)_{k\geq 0}$ is defined by $\mathbf{b} = \mathbf{a}\,C^{-1}$.

Proof. The defining relation $\mathbf{b} = \mathbf{a}\,C^{-1}$ yields $\mathbf{a} = \mathbf{b}\,C$ and, after multiplication by the column vector $\mathbf{x}$,

$$\mathbf{a}\,\mathbf{x} = \mathbf{b}\,C\,\mathbf{x}.$$

Making use of (1), this matrix identity translates into

$$\sum_{k\geq 0} a_k x^k = \sum_{k\geq 0} b_k x^k C(x)^{k+1}.$$

Now apply Lemma 1 with $f(x) = C(x)$. □

Proposition 3. *For a nonnegative sequence $\mathbf{a} = (a_k)_{k\geq 0}$, its run transform is nonnegative if and only if the sequence $\mathbf{x} := \mathbf{a}\,C^{-1}$ is nonnegative.*

Proof. We have $C(x,0) = (1-x)C\big(x(1-x)\big) = 1$ and it follows from Proposition 2 that column 0 of the run transform, given by $F(x,0)$, is (the transpose of) $\mathbf{x}$. So the condition is certainly necessary. Sufficiency will follow if we know that each power of $C(x,y)$ is the generating function of a nonnegative array. For $C(x,y)$ itself, nonnegativity follows from a combinatorial interpretation in terms of decorated forests [2, Section 9] or from Lemma 4, but we can also give an analytic proof as follows. Say $(u_{i,j})_{i\geq 0,\ 0\leq j\leq i}$ is the array of coefficients for $C(x,y)$. We have the identity $(2xC(x,y)-1)^2 = 1-4x(1-x)/(1-xy)$, leading to

$$xC(x,y)^2 = C(x,y) - \frac{1-x}{1-xy}.$$

Picking out coefficients leads to a recurrence for $u_{i,j}:\ u_{0,0} = 1$, and

$$u_{n,k} = \begin{cases} \displaystyle\sum_{i=0}^{n-1}\sum_{j=0}^{n} u_{i,j}u_{n-1-i,n-j} + 1, & \text{if } 1\leq k\leq n; \\ \displaystyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} u_{i,j}u_{n-1-i,n-1-j} - 1, & \text{if } 0\leq k = n-1; \\ \displaystyle\sum_{i=0}^{n-1}\sum_{j=0}^{k} u_{i,j}u_{n-1-i,k-j}, & \text{if } 0\leq k\leq n-2, \end{cases}$$

from which it is easy to see that $u_{i,j}$ is nonnegative, the -1 in the middle equality notwithstanding. Finally, it is easy to check that nonnegativity of $C(x,y)$ implies nonnegativity of all its powers. □

4. The run transform of the Catalan numbers

Lemma 4. *The run transform $C(x,y)$ of the Catalan number generating function $C(x)$ counts Dyck paths by size and number of pyramid ascents, equivalently, noncrossing partitions by size and number of runs.*

Proof. Let $F(x,y)$ denote the generating function for Dyck paths by size and number of pyramid ascents. A Dyck path P is either empty or has the decomposition $P = U^r DP_1DP_2D\cdots DP_r$ for some $r\geq 1$, where the P_i are Dyck paths. Each pyramid ascent in $P_1,\dots,P_r$ is a pyramid ascent in P and, if $P_1,\dots,P_{r-1}$ are all empty paths, then the first ascent of P is also a pyramid ascent, contributing a y factor. We thus obtain

$$F = 1 + \sum_{r\geq 1} x^r\big(y + F^{r-1} - 1\big)F$$

which leads at once to

$$xF^2 - F + \frac{1-x}{1-xy} = 0,$$

an equation whose solution is $F(x, y) = C(x, y)$. □

Recall that $\mathcal{N}_k$ is the set of nonnegative U–D paths with k more downsteps than upsteps.

Theorem 5. *The run transform of the generating function for $\mathcal{N}_k$ by size counts $\mathcal{N}_k$ by size and number of pyramid ascents.*

Proof. A path in $\mathcal{N}_k$ decomposes as $P_1DP_2D \cdots P_kDP_{k+1}$ with each P_i a Dyck path. The statistics size and number of pyramid ascents are additive over this decomposition. So one multiplies the generating functions given by Lemma 4, and the generating function for $\mathcal{N}_k$ by size and number of pyramid ascents is $x^kC(x, y)^{k+1}$. By Lemma 1, the run transform of $x^kC(x)^{k+1}$ is $x^kC(x, y)^{k+1}$. □

In subsequent sections we generalize this result in 3 ways: (i) from Dyck paths to U–D paths in which each ascent has length divisible by j, (ii) from Dyck paths to U–F-D paths, where flatsteps $F = (1, 0)$ are allowed, and (iii) from noncrossing partitions to run-closed families of s-partitions, defined in Section 6.

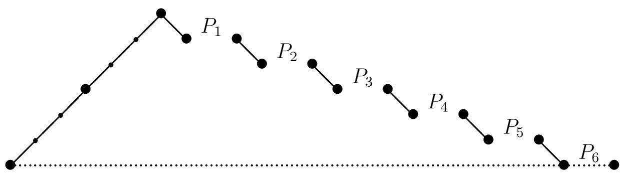
5. U^j-D paths

Fix a positive integer j. A *j-Dyck* path is a Dyck path in which each ascent has length divisible by j. Equivalently, it can be viewed as a nonnegative lattice path of the so-called j-upsteps (j, j) and (ordinary) downsteps $(1, -1)$. Its *size* is the number of j-upsteps, equivalently, (number of downsteps)$/j$. A *j-nice* pyramid ascent is one that ends at height $\equiv 0 \pmod{j}$.

Lemma 6. *The run transform of the generating function for j-Dyck paths by size is the generating function for j-Dyck paths by size and number of pyramid ascents.*

Proof. Let $F_0(x, y)$ denote the generating function for j-Dyck paths with x marking size and y marking the number of j-nice pyramid ascents. To find an equation for $F_0(x, y)$, it is convenient to introduce the generating function $F_i(x, y)$, i an integer, for the number of pyramid ascents that end at height $\equiv i \pmod{j}$. Set $F = F_0F_1 \cdots F_{j-1}$.

A nonempty j-Dyck path P has a decomposition as illustrated for $j = 3$ and $r = 2$ where r is the number of initial j-upsteps, and the P_i's are all j-Dyck paths (possibly empty).



A j-Dyck path with $j = 3$

In general, the decomposition is

$$U^{jr}DP_1DP_2 \cdots DP_{jr},$$

for some $r \geq 1$. To obtain an expression for F_0 from this decomposition, P_1 contributes F_1 because it starts at height $\equiv -1$ mod j, P_2 contributes $F_2, \ldots, P_j$ contributes F_0, P_{j+1} contributes F_1, and so on, cyclically. And if $P_1, P_2, \ldots, P_{jr-1}$ are all empty, the first ascent of P is a pyramid ascent. Splitting into the two cases where $P_1, P_2, \ldots, P_{jr-1}$ are all empty or not, we thus obtain

$$F_0 = 1 + \sum_{r\geq 1} x^r \big(y + F_1F_2 \cdots F_{j-1}F^{r-1} - 1\big)F_0$$

which simplifies to

$$F_0 = 1 + (y-1)F_0\frac{x}{1-x} + \frac{xF}{1-xF}. \tag{3}$$

For $i \not\equiv 0 \pmod{j}$, there is no need to split into cases. We find $F_i = 1 + xF + x^2F^2 + \cdots$, leading to

$$F_i = \frac{1}{1-xF} \quad \text{for } i \not\equiv 0 \pmod{j}. \tag{4}$$

Eliminating f from (3) and (4), we obtain

$$F_i(x,y) = \frac{1-xy}{1-x} F_0(x,y) \quad \text{for } i \not\equiv 0 \pmod{j}. \tag{5}$$

Hence,

$$F = \left(\frac{1-xy}{1-x}\right)^{j-1} F_0^j$$

and (3) becomes, after further simplification

$$(1-x)^j - (1-x)^{j-1}(1-xy)F_0 + x(1-xy)^j F_0^{j+1} = 0. \tag{6}$$

From (6), $f(x) := F_0(x,1)$ has a defining equation $1 - f + xf^{j+1} = 0$, and the run transform of $f(x)$ satisfies (6). Hence the run transform of $f(x)$ is $F_0(x,y)$. $\square$

Now fix nonnegative integers m and d. A (j,m,d)-U-D path is a path of j-upsteps and downsteps that starts (for convenience) at $(0,m)$, has lowest point at level $-d$, and ends on the x-axis. A $(j,0,0)$-U-D path is just a j-Dyck path. The size of a (j,m,d)-U-D path is $\lfloor(\text{number of downsteps})/j\rfloor$, and so it is convenient to express m as $jk+\ell$ with $0 \le \ell \le j-1$.

Theorem 7. *The run transform of the generating function for (j,m,d)-U-D paths by size is the generating function for (j,m,d)-U-D paths by size and number of pyramid ascents.*

Proof. Let $G(x,y)$ denote the generating function for (j,m,d)-U-D paths with x marking size and y marking the number of j-nice pyramid ascents. A (j,m,d)-U-D path has the decomposition:

$$P_0\, D\, P_1\, D\, P_2, \ldots, D\, P_{m+d}\, U^j\, P'_{m+d-1}\, U^j\, P'_{m+d-2}, \ldots, U^j\, P'_m. \tag{7}$$

where the P_i and the P'_i are all j-Dyck paths.

If a P'_i begins with a pyramid, the pyramid is killed by the immediately preceding U^j. This necessitates introducing the generating function $H(x,y)$ for j-Dyck paths that begin with a pyramid:

$$H(x,y) = \sum_{i\ge 1} x^i y F_0 = \frac{xy}{1-x} F_0.$$

The decomposition (7) together with (5) yields

$$\begin{aligned} G(x,y) &= x^{k+d} F_{-m}, \ldots, F_{-1}\, F_0\, F_1\, F_2, \ldots, F_{jd} \left(\frac{H}{y} + F_0 - H\right) \\ &= x^{k+d} F_0^{k+d+1} F_1^{m-k+(j-1)d} \left(\frac{1-xy}{1-x} F_0\right)^d \\ &= x^{k+d} \left(\frac{1-xy}{1-x}\right)^{m-k+jd} F_0^{m+1+(j+1)d}. \end{aligned}$$

Hence $g(x) := G(x,1) = x^{k+d} F_0(x,1)^{m+1+(j+1)d}$ is the generating function for (j,m,d)-U-D paths by size. The run transform of $g(x)$ is

$$\begin{aligned} \frac{1-x}{1-xy}\, g\left(\frac{x(1-x)}{1-xy}\right) &= \frac{1-x}{1-xy} \left(\frac{x(1-x)}{1-xy}\right)^{k+d} F_0\left(\frac{x(1-x)}{1-xy}, 1\right)^{m+1+(j+1)d} \\ &= x^{k+d} \left(\frac{1-xy}{1-x}\right)^{m+jd-k} \left(\frac{1-x}{1-xy} F_0\left(\frac{x(1-x)}{1-xy}, 1\right)\right)^{m+1+(j+1)d} \\ &= x^{k+d} \left(\frac{1-xy}{1-x}\right)^{m+jd-k} F_0(x,y)^{m+1+(j+1)d} \\ &= G(x,y), \end{aligned}$$

the next to last equality using the fact that the run transform of $F_0(x,1)$ is $F_0(x,y)$ (Lemma 6). $\square$

6. Run-closed families of s-partitions

A *set-of-lists* partition, or *s-partition* for short, also known as a *fragmented permutation* [4, p. 125], is a partition π of a set S into a set of lists. The size of π, denoted $|\pi|$, is $|S|$. An s-partition is *standard* if its support set is an initial segment of the positive integers. We use the familiar term *blocks* for the lists in an s-partition, and we always arrange the blocks in increasing order of their first entry. Recall that a run is a block that consists of consecutive integers in increasing order. Thus the s-partition 3 8 1 / 4 5 6 / 7 2 / 9 has size 9 and four blocks, two of which are runs, 4 5 6 and 9. A permutation can be viewed as an s-partition via its disjoint cycle decomposition; for definiteness, we define a *cycle* to be a list whose smallest entry occurs first.

To *delete* a run from a standard s-partition means to remove it and standardize what remains (replace smallest entry by 1, second smallest by 2, and so on). Thus deleting the run 23 from 178/23/465/9 yields 156/243/7. To *insert* a run $i+1, \dots, i+j$ into a standard s-partition π means to increment by j all elements of π that exceed i and adjoin $i+1, i+2, \dots, i+j$ as a new block. The result will be a standard s-partition provided $0 \le i \le |\pi|$. For example, inserting the run 456 into 15/342 yields 18/372/456. When runs are successively deleted from an s-partition, the order of deletion is immaterial and the result is always the same run-free s-partition. For 178/23/465/9, the result is 156/243.

Let $\mathcal{P}$ denote the set of all standard s-partitions, including the empty one ϵ.

Definition 8. A *run-closed* family of s-partitions is a subset of $\mathcal{P}$ that is closed under insertion and deletion of runs.

To present a few examples of run-closed families, let us recall some terminology and facts for ordinary set partitions. A set partition is an s-partition in which each block is an increasing list $u_1 < u_2 < \cdots < u_k$, and it can be represented graphically as the numbers $1, 2, \dots, n$ arranged in order around a circle with a line joining each pair of nearest-neighbor entries, (u_i, u_{i+1}), in each non-singleton block. It is noncrossing if no two of these lines cross. Similarly, a set partition is nonoverlapping if the lines joining the smallest and largest entries of each block are noncrossing. The noncrossing partitions of $[n]$ form a subset of the nonoverlapping partitions of $[n]$, and a proper subset if $n \ge 5$. For example, 135/24 is nonoverlapping but fails to be noncrossing. A set partition is nonnesting if for each nearest-neighbor pair (u, v) in a non-singleton block and nearest-neighbor pair (x, y) in a different non-singleton block, the interval $[u, v]$ does not contain the interval $[x, y]$. For example, 13/24 is crossing but nonnesting, while 14/23 is nesting but noncrossing. Both noncrossing and nonnesting partitions of $[n]$ are counted by the Catalan number C_n.

Here are some examples of run-closed families and, where available, their counting sequences.

- the family $\mathcal{P}$ itself [3], A000262
- set partitions, A000110
- noncrossing s-partitions, A088368
- nonoverlapping s-partitions
- permutations, via the disjoint cycle decomposition, A000142
- the intersection of any collection of run-closed families.

The run-closed property for both noncrossing and nonoverlapping partitions is evident from the above lines-on-a-circle representation. On the other hand, the family of nonnesting partitions is not run-closed. While closed under deletion of runs, indeed under deletion of arbitrary blocks, it is not closed under insertion of runs. For example, inserting the run 23 into the one-block nonnesting partition 12 produces the nesting partition 14/23.

Now we can state our result for s-partitions.

Theorem 9. *Let $\mathcal{F}$ be a run-closed family of s-partitions with size generating function $f(x)$. Then the run transform $F(x, y)$ of $f(x)$ counts $\mathcal{F}$ by size and number of runs.*

The proof of this theorem and the necessary preliminaries occupy the remainder of this section.

A run-closed family $\mathcal{F}$ of s-partitions is determined by its *run-free* members. This is because all members of $\mathcal{F}$ can be obtained by successively inserting runs into its run-free members. We call the set of run-free s-partitions in a run-closed family $\mathcal{F}$ the *basis* of $\mathcal{F}$.

Every set of run-free s-partitions in $\mathcal{P}$ serves as a basis for a run-closed family of s-partitions. We have the following two easily proved results for ordinary partitions.

Lemma 10. *A standard noncrossing partition is either empty or contains a run.*

Corollary 11. *The singleton set consisting of the empty partition is the basis for the family of noncrossing partitions.*

Every s-partition can be successively pruned of runs from right to left, leaving a run-free s-partition (possibly empty) and a sequence of runs, its *run-deletion sequence*, from which the original s-partition can be recovered, as illustrated.

Current s-partition	Deleted run
1 12 10 / 2 6 8 / 3 / 4 5 / 7 / 9 11	7
1 11 9 / 2 6 7 / 3 / 4 5 / 8 10	4 5
1 9 7 / 2 4 5 / 3 / 6 8	3
1 8 6 / 2 3 4 / 5 7	2 3 4
1 5 3 / 2 4	
Run-free s-partition = 1 5 3 / 2 4, run-deletion sequence = (2 3 4, 3, 4 5, 7)	

Furthermore, the number of runs in the original s-partition is captured in the run-deletion sequence as the number of runs that are disjoint from their immediate predecessor. (The first run vacuously meets this condition.) This is because, in reconstructing the s-partition, when a new run is inserted, the only existing run that it can destroy is its predecessor run (if present) and it will do so precisely when it overlaps its predecessor. A run-deletion sequence is of course specified by the first entries and lengths of its members, say $(a_i)_{i=1}^r$ and $(\ell_i)_{i=1}^r$ in reverse order of deletion. In the example $(a_i)_{i=1}^4 = (2, 3, 4, 7)$; $(\ell_i)_{i=1}^4 = (3, 1, 2, 1)$.

For a run-closed family $\mathcal{F}$ of s-partitions, let $\mathcal{F}(n) = \{\rho \in \mathcal{F} : |\rho| = n\}$, the members of $\mathcal{F}$ of size n.

Proposition 12. *Fix a run-free s-partition π of size k. Let $\mathcal{F}$ denote the set of s-partitions that prune to π, and suppose $n > k$. Then the run-deletion sequences of s-partitions in $\mathcal{F}(n)$, as specified by $(a_i)_{i=1}^r$ and $(\ell_i)_{i=1}^r$, are characterized by the following conditions:*

$$
\begin{aligned}
&r \geq 1 \text{ and all } a\text{'s and } \ell\text{'s are positive integers},\\
&k + \ell_1 + \ell_2 + \cdots + \ell_r = n,\\
&a_1 < a_2 < \cdots < a_r,\\
&a_1 \leq k + 1,\\
&a_2 \leq k + \ell_1 + 1,\\
&a_3 \leq k + \ell_1 + \ell_2 + 1,\\
&\vdots\\
&a_r \leq k + \ell_1 + \ell_2 + \cdots + \ell_{r-1} + 1.
\end{aligned}
$$

Proof. The first two conditions are obvious. Now, when a run is deleted, the result is still a standard s-partition. Clearly, $a_r + \ell_r \leq n + 1$ and so $a_r \leq n - \ell_r + 1 = k + \ell_1 + \ell_2 + \cdots + \ell_{r-1} + 1$, and similarly for the other inequalities. Because runs are deleted right to left, we get $a_1 < a_2 < \cdots < a_r$.

Conversely, when runs are inserted successively into the run-free s-partition π to build up members of $\mathcal{F}(n)$, the runs are arbitrary subject only to the conditions that the run currently being inserted begins at an integer no larger than 1 + the size of the s-partition into which it is being inserted, for otherwise there would be a gap and the resulting s-partition would not have an initial segment of the positive integers as support. □

Now we can establish

Proposition 13. *Fix a run-free standard s-partition π. The number of s-partitions of given size and run count that prune to π depends only on the size of π, not on its actual blocks.*

Proof. Suppose $\mathcal{G}_1$ is a run-closed family all of whose members prune to π_1 and $\mathcal{G}_2$ is a run-closed family all of whose members prune to π_2. Suppose further that π_1 and π_2 have the same size k. We wish to show that $|\mathcal{G}_1(n)| = |\mathcal{G}_2(n)|$ for all $n > k$ (it is obviously true for $n = k$). Since the characterization of the run-deletion sequences for $\mathcal{G}_1(n)$ makes no reference to π_1 other than through its size, this equality follows from two observations:

(i) members of $\mathcal{G}_i(n)$ are in one-to-one correspondence with their run-deletion sequences, $i = 1, 2$;

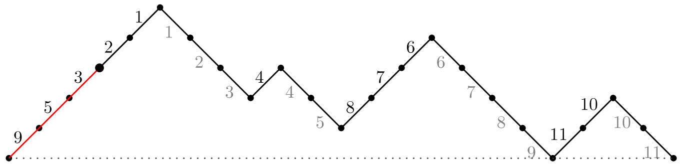
(ii) the set of run-deletion sequences for $\mathcal{G}_1(n)$ is *identical* with that for $\mathcal{G}_2(n)$: both sets are characterized by the conditions of Proposition 12. □

Now we consider a canonical singleton basis of each size k, say the one-block s-partition $k, k-1, \ldots, 2, 1$ (the empty s-partition if $k = 0$), and let $\mathcal{F}_k$ denote the run-closed family generated by it. A technical difficulty arises if $k = 1$ because the only s-partition of size 1 is the one-block s-partition, (1), which is a run. (A workaround would be to color it red, maintain the color when runs are inserted – inserting 12 into 1 yields 12/3 – and declare that red entries are not to be considered runs.)

By Corollary 11, $\mathcal{F}_0$ is the family of noncrossing ordinary partitions, and $\mathcal{F}_k$ can be characterized for general k: it consists of the noncrossing s-partitions in which (i) all blocks are increasing except for one block of length k which is decreasing and (ii) this decreasing block is not covered by entries in another block, in other words, there is no increasing block containing integers $a < b$ such that all entries of the decreasing block lie in the interval $[a, b]$.

For $k \geq 2$, a slight modification of the standard bijection gives a bijection from nonnegative paths that start at $(0, k)$ to $\mathcal{F}_k$. First, prepend k upsteps to turn the nonnegative path into a Dyck path.

Apply the standard bijection of Section 2, as illustrated in an example with $k = 3$. The blocks are in the natural order and each block is decreasing.



$$\overbrace{9\ 5\ 3}^{k}\ 2\ 1 \qquad 4 \qquad 8\ 7\ 6 \qquad 11\ 10 \qquad \rightarrow$$

$$9\ 5\ 3 \qquad 1\ 2 \qquad 4 \qquad 6\ 7\ 8 \qquad 10\ 11 \qquad \rightarrow$$

$$1\ 2 \qquad 4 \qquad 6\ 7\ 8 \qquad 9\ 5\ 3 \qquad 10\ 11$$

The first block will have length $\geq k$ and will end with 1, and all blocks will be decreasing. Split the first block after the k-th entry into 2 blocks. Reverse all blocks except the new first block, and transfer the new first block to the appropriate position so that first entries are increasing. The resulting s-partition is a member of $\mathcal{F}_k$ and the mapping is reversible.

This correspondence preserves size and identifies runs with pyramid ascents. So we have the following proposition.

Proposition 14. *The generating function for $\mathcal{F}_k$ by size and number of runs is $x^k C(x, y)^{k+1}$.*

The linearity of the run transform, along with Propositions 13 and 14, now yields the following.

Proposition 15. *Let $\mathcal{F}$ be any run-closed family of s-partitions containing a_k (≥ 0) run-free s-partitions of size k, $k \geq 0$. Then $F(x, y) := \sum_{k \geq 0} a_k x^k C(x, y)^{k+1}$ is the generating function to count $\mathcal{F}$ by size and number of runs.* □

Proof of Theorem 9. In the notation of Proposition 15, the generating function by size of the run-closed family $\mathcal{F}$ is

$$f(x) = F(x, 1) = \sum_{k \geq 0} a_k x^k C(x)^{k+1},$$

since $C(x, 1) = C(x)$. Lemma 1 then yields that the run transform of $f(x)$ is indeed $F(x, y)$. □

Remark. Theorem 9 applied to the case of set partitions implies that the bivariate generating function for set partitions according to size and number of runs is $\sum_{n \geq 0} B_n x^n (1-x)^{n+1}/(1-xy)^{n+1}$, where B_n are the Bell numbers. In particular, the generating function for the number of partitions such that no block is a run is $(1-x) \sum_{n \geq 0} B_n \big(x(1-x)\big)^n$ [6, Exercise 111, pp. 137,192–193].

7. Generalization of Theorem 9

Fix a positive integer j. A j-compatible s-partition is one in which each block has length divisible by j. Define its size to be n/j where n is the cardinality of its support set. A j-compatible run (j-run for short) is one whose length and last entry are both divisible by j. A j-run-closed family of j-compatible s-partitions is one that is closed under insertion and deletion of j-runs.

Theorem 16. *Let $\mathcal{F}$ be a j-run-closed family of j-compatible s-partitions with size generating function $f(x)$. Then the run transform $F(x, y)$ of $f(x)$ counts $\mathcal{F}$ by size and number of j-runs.*

Proof. The "j" analogue of $\mathcal{F}_k$ is the family of j-compatible s-partitions with singleton basis $(jk, jk-1, \ldots, 1)$, which corresponds under the standard bijection to the family of $(j, jk, 0)$-U-D paths. This bijection preserves size and identifies j-runs with j-nice pyramid ascents. Apply Theorem 7. □

As an example, we have the following result.

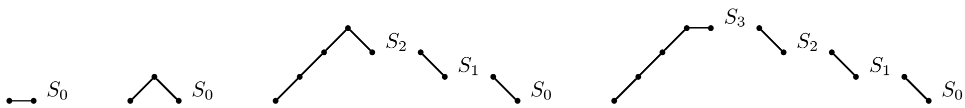
Corollary 17. *If $f(x)$ denotes the generating function for permutations of $[2n]$ in which each cycle has even length (A001818) by size n, then the run transform of f counts these permutations by size and by number of cycles that consist of consecutive integers ending at an even integer.*

8. $U-F-D$ paths

Fix nonnegative integers m and d and consider the class $\mathcal{A}_{m,d}$ of lattice paths of upsteps $U = (1, 1)$, downsteps $D = (1, -1)$, and flatsteps $F = (2, 0)$ that start at $(0, m)$, end on the x-axis and that reach lowest level $-d$, with size measured by (number of flatsteps) + (number of downsteps). Thus $\mathcal{A}_{0,0}$ is the class of Schröder paths with the usual measure of size. The definition of pyramid ascent carries over verbatim to $\mathcal{A}_{m,d}$.

Lemma 18. *Let $F(x, y, z_0, z_1, z_2, \ldots)$ denote the generating function for Schröder paths with x marking size, y marking number of pyramid ascents, and z_i marking number of flatsteps at level i, $i \geq 0$. Thus $f(x, z\text{'s}) := F(x, 1, z\text{'s})$ is the generating function disregarding pyramid ascents. Then the run transform of f is F.*

Proof. Set $F_j = F(x, y, z_j, z_{j+1}, z_{j+2}, \ldots)$. Thus $F_0 = F$. A Schröder path is either empty or, by considering the first non-upstep, begins with one of the prefixes F, UD, U^rD $(r \geq 2)$, U^rF $(r \geq 1)$. Thus a nonempty Schröder path has precisely one of the following forms, illustrated for $r = 3$, where the S_i $(i \geq 0)$ denote arbitrary Schröder paths.



decompositions of nonempty Schröder paths

From this schematic picture, we see that

$$F_0 = 1 + xz_0F_0 + xyF_0 + \left(\sum_{r\geq 2}(F_{r-1}\ldots F_1 - 1)F_0 + \sum_{r\geq 2} x^r yF_0\right) + \sum_{r\geq 1} x^{r+1} z_r F_r F_{r-1}\ldots F_0$$

from which we obtain by routine manipulation

$$\frac{1-xy}{1-x}F_0 = 1 + \sum_{r\geq 1} x^r(1+z_{r-1})F_{r-1}F_{r-2}\ldots F_0, \tag{8}$$

a recursion for $F = F_0$ (bear in mind that $F_1, F_2, \ldots$ are merely abbreviations for functions derived from F). This recursion has a unique solution for F because it determines the constant term and then the coefficients of $x, x^2, \ldots$ in turn.

Set $f_j(x, z_j, z_{j+1}, \ldots) = F_j(x, 1, z_j, z_{j+1}, \ldots)$. Thus $f_0 = f$. From (8) with $y = 1$, we have

$$f_0 = 1 + \sum_{r\geq 1} x^r(1+z_{r-1})f_{r-1}f_{r-2}\ldots f_0. \tag{9}$$

The run transform of $f_0(x, z_0, z_1, \ldots)$ is

$$H_0(x, y, z_0, z_1, \ldots) := \frac{1-x}{1-xy} f_0\left(\frac{x(1-x)}{1-xy}, z_0, z_1, \ldots\right),$$

and we define H_j, $j \geq 1$ by relabeling z indices just as for F_j. To verify that H_0 and F_0 are equal, replace x by $\frac{x(1-x)}{1-xy}$ in (9) to obtain

$$\begin{aligned}&\frac{1-xy}{1-x}\left(\frac{1-x}{1-xy} f_0\left(\frac{x(1-x)}{1-xy}, z_0, z_1, \ldots\right)\right)\\ &\quad = 1 + \sum_{r\geq 1} x^r(1+z_{r-1})\frac{1-x}{1-xy} f_{r-1}\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right)\cdots\frac{1-x}{1-xy} f_0\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right)\end{aligned}$$

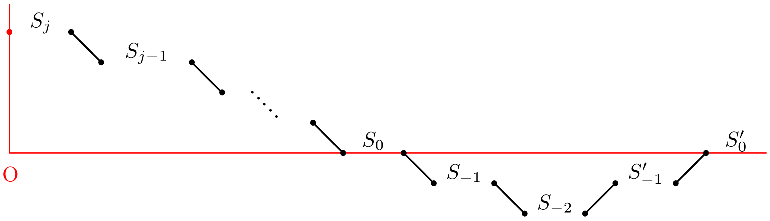
or

$$\frac{1-xy}{1-x}H_0 = 1 + \sum_{r\geq 1} x^r(1+z_{r-1})H_{r-1}H_{r-2}\ldots H_0. \tag{10}$$

Comparing (8) and (10) we see that $H_0 = F_0$ because, as noted above, (8) has a unique solution. □

Theorem 19. *Let $G(x, y, z_{-d}, z_{-(d-1)}, \ldots, z_0, z_1, z_2, \ldots)$ denote the generating function for $\mathcal{A}_{m,d}$ with x marking size, y marking number of pyramid ascents, and z_i marking number of flatsteps at level i. Thus $g(x, z\text{'s}) := G(x, 1, z\text{'s})$ is the generating function disregarding pyramid ascents. Then the run transform of g is G.*

Proof. A path in $\mathcal{A}_{j,d}$ has the form below, illustrated for $d = 2$, where S_i and S'_i denote Schröder paths.



decomposition of path in $\mathcal{A}_{j,d}$

Consequently,

$$G = x^j F_j F_{j-1} \ldots F_0 + \sum_{k=1}^{d} x^{j+k} F_j F_{j-1} \ldots F_0 F_{-1} \ldots F_{-k} \prod_{i=0}^{k-1} \left(\frac{H_{-i}}{y} + (F_{-i} - H_{-i}) \right), \tag{11}$$

where $H_0(x, y, z_0, z_1, \ldots)$ is the generating function for nonempty Schröder paths that start with a pyramid, and H_{-i} $(i \geq 1)$ is obtained from H_0 by relabeling z indices in the usual way. The introduction of H_0 is necessary because if a Schröder path S'_{-i} $(i \geq 0)$ begins with a pyramid, then the immediately preceding upstep kills the initial pyramid ascent in S'_{-i}. Clearly,

$$H_0(x, y, z_0, z_1, \ldots) = \sum_{k \geq 1} x^k y F = \frac{xyF}{1-x}. \tag{12}$$

Now let $g(x, z_{-d}, z_{-(d-1)}, \ldots, z_0, \ldots) = G(x, 1, z_{-d}, z_{-(d-1)}, \ldots, z_0, \ldots)$. Thus

$$g = x^j f_j f_{j-1} \ldots f_0 + \sum_{k=1}^{d} x^{j+k} f_j f_{j-1} \ldots f_0 f_{-1} \ldots f_{-k} \prod_{i=0}^{k-1} f_{-i}. \tag{13}$$

From (12), we have

$$\frac{H_{-i}}{y} + (F_{-i} - H_{-i}) = \frac{1-xy}{1-x} F_{-i}. \tag{14}$$

Using (13), the run transform of g is

$$\begin{aligned}
&\frac{1-x}{1-xy} g\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \\
&= x^j \frac{(1-x)^{j+1}}{(1-xy)^{j+1}} f_j\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \ldots f_0\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \\
&\quad + \sum_{k=1}^{d} x^{j+k} \frac{(1-x)^{j+k+1}}{(1-xy)^{j+k+1}} f_j\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) f_{j-1}\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \ldots f_0\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \\
&\quad \times f_{-1}\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \ldots f_{-k}\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \prod_{i=0}^{k-1} f_i\left(\frac{x(1-x)}{1-xy}, z\text{'s}\right) \\
&= x^j F_j \ldots F_0 + \sum_{k=1}^{d} x^{j+k} F_j F_{j-1} \ldots F_0 F_{-1} \ldots F_{-k} \prod_{i=0}^{k-1} \frac{1-xy}{1-x} F_{-i},
\end{aligned}$$

which, in view of (14), is the same expression as in (11). The run transform of g is thus G. $\square$

9. Concluding remark

We believe there is a version of our results that includes both j-upsteps and flatsteps. The setting is paths of j-upsteps $U_j = (j, j)$, flatsteps $F = (2, 0)$, and downsteps $D = (1, -1)$, that start at $(0, m)$ and end on the x-axis; j and m are nonnegative integers. In this generality, the size of a path is $\lfloor$(number of D's + number of F's)/$j\rfloor$ (so it does not matter whether we consider flatsteps to be of length 1 or 2). Furthermore, the "nice" pyramid ascents to count are those whose endpoint (a, b) has a divisible by j rather than b divisible by j; that is, the abscissa rather than the ordinate of the endpoint is divisible by j. These conditions are equivalent if $j = 1$ or if there are no flatsteps and m is divisible by j.

Conjecture 20. *Fix nonnegative integers j and m. Let* $G(x, y, z_{-d}, z_{-(d-1)}, \ldots, z_0, z_1, z_2, \ldots)$ *denote the generating function for* U_j*-F-D paths with x marking size, y marking number of ''nice'' pyramid ascents, and* z_i *marking number of flatsteps at level i. Thus* $g(x, z\text{'s}) := G(x, 1, z\text{'s})$ *is the generating function disregarding pyramid ascents. Then the run transform of g is G.*

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